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On the Almost Everywhere Convergence of Fejér Means of Functions on the Group of 2-Adic Integers

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In the present paper we prove the almost everywhere convergence of the Fejér means of integrable functions on the group of 2-adic integers.

1. INTRODUCTION AND RESULTS

We follow the standard notions of dyadic analysis introduced by the mathematicians F. Schipp, P. Simon, and W. R. Wade (see e.g. [Sch]) and others. Denote the set of natural numbers by $\mathbf{N} := \{0, 1, ...\}$, the set of positive integers by $\mathbf{P} := \mathbf{N} \setminus \{0\}$, and the unit interval by $I := [0, 1)$. Denote by $\lambda(B) = |B|$ the Lebesgue measure of the set $B (B \subset I)$. Denote by $L^p(I)$ the usual Lebesgue spaces and by $\|\cdot\|_p$ the corresponding norms $(1 \leqslant p \leqslant \infty)$. Set

$$\mathscr{I} := \left\{ \left[\frac{p}{2^n}, \frac{p+1}{2^n} \right) : p, n \in \mathbf{N} \right\},$$

the set of dyadic intervals, and for given $x \in I$ let $I_n(x)$ denote the interval $I_n(x) \in \mathscr{I}$ of length 2^{-n} which contains $x (n \in \mathbf{N})$. Also use the notion $I_n := I_n(0) (n \in \mathbf{N})$. Let

$$x = \sum_{n=0}^{\infty} x_n 2^{-(n+1)},$$

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the dyadic expansion of $x \in I$, where $x_n = 0$ or 1 and if x is a dyadic rational number ($x \in \{p/2^n : p, n \in \mathbf{N}\}$) we choose the expansion which terminates in 0's. The 2-adic (or arithmetic) sum $a + b := \sum_{n=0}^{\infty} r_n 2^{-(n+1)}$ $(a, b \in I)$, where bits $q_n, r_n \in \{0, 1\}$ $(n \in \mathbf{N})$ are defined recursively as follows: $q_{-1} := 0$, $a_n + b_n + q_{n-1} = 2q_n + r_n$ for $n \in \mathbf{N}$. (Since q_n, r_n take on only the values 0, 1, these equations uniquely determine the coefficients q_n and r_n.) The group $(I, +)$ is called the group of 2-adic integers. Set

$$\varepsilon(t) := \exp(2\pi \imath t) \qquad (t \in \mathbf{R}),$$

where $\imath = (-1)^{1/2}$. Set

$$v_{2^n}(x) := \varepsilon\left(\frac{x_n}{2} + \cdots + \frac{x_0}{2^{n+1}}\right) \qquad (x \in I,\ n \in \mathbf{N})$$

and

$$v_n := \prod_{n=0}^{\infty} v_{2^j}^{n_j},$$

where $\mathbf{N} \ni n = \sum_{i=0}^{\infty} n_i 2^i$ $(n_i \in \{0, 1\}$ $(i \in \mathbf{N}))$. It is known [Hew] that the system $(v_n, n \in \mathbf{N})$ is the character system of $(I, +)$. Denote by $\hat{f}(n) := \int_I f \bar{v}_n \, d\lambda$ $(n \in \mathbf{N})$, $D_n := \sum_{k=0}^{n-1} v_k$, and $K_n := (1/n) \sum_{k=1}^{n-1} D_k$ $(n \in \mathbf{P})$ the Fourier coefficients, the Dirichlet kernels, and the Fejér kernels, respectively. It is also known that

$$\begin{aligned}
\sigma_n f(y) &= \int_I f(x) \, K_n(y + x) \, d\lambda(x) = \frac{1}{n} \sum_{k=1}^{n} S_k f(y) \\
&= \frac{1}{n} \sum_{k=1}^{n} \int_I f(x) \, D_k(y + x) \, d\lambda(x) \\
&= \frac{1}{n} \sum_{k=1}^{n} \sum_{j=0}^{k-1} \left(\int_I f(x) \, \bar{v}_j(x) \, d\lambda(x) \right) v_j(y) \\
&= \frac{1}{n} \sum_{k=1}^{n} \sum_{j=0}^{k-1} \hat{f}(j) \, v_j(y) \qquad (n \in \mathbf{P},\ y \in I).
\end{aligned}$$

We prove

THEOREM 1. *$\sigma_n f \to f$ a.e. $(f \in L^1(I))$.*

Theorem 1 for the general case (for the Fejér means the group of p-adic integers) is the conjecture of M. H. Taibleson [Tai, p. 114.]. Theorem 1 with respect to the Walsh–Paley system on the Walsh group is proved by Fine [Fin]. This result with respect to the p-series fields is due to Taibleson

[Tai2]. Later, this result was generalized for the so-called bounded Vilenkin groups with respect to the Vilenkin system by Pál and Simon [P-S]. The noncommutative case is discussed by the author [Gát].

In order to prove Theorem 1 we need

LEMMA 2 ([S-W]).

$$D_n(z)=v_n(z)\sum_{j=0}^{\infty} n_j(-1)^{z_j} D_{2^j}(z),$$

where

$$D_{2^k}(z)=\begin{cases}2^k & \text{if } z\in I_k\\ 0 & \text{if } z\notin I_k\end{cases}\qquad (n\in \mathbf{P},\ k\in \mathbf{N},\ z\in I).$$

LEMMA 3. *Let $A\geqslant \tau$ be fixed natural numbers. Then,*

$$\int_{I_\tau\setminus I_{\tau+1}} \sup_{N\geqslant 2^A} |K_N(z)|\, d\lambda(z)\leqslant c(2^{\tau-A})^{1/2}.$$

Throughout this work $c>0$ will denote an absolute constant which will not necessarily be the same at different occurrences.

The so-called dyadic Hardy space H is defined as follows [Sch, Sim]. A function $a\in L^\infty(I)$ is called an atom, if either $a=1$ or a has the following properties: $\operatorname{supp} a\subseteq I_a$, $\|a\|_\infty\leqslant |I_a|^{-1}$, $\int_I a=0$, for some $I_a\in \mathscr{I}$. We say that the function f belongs to H, if f can be represented as $f=\sum_{i=0}^\infty \lambda_i a_i$, where a_i's are atoms and, for the coefficients λ_i $(i\in \mathbf{N})$, $\sum_{i=0}^\infty |\lambda_i|<\infty$ is true. It is known that H is a Banach space with respect to the norm

$$\|f\|_H:=\inf \sum_{i=0}^{\infty} |\lambda_i|,$$

where the infimum is taken over all decompositions $f=\sum_{i=0}^\infty \lambda_i a_i\in H$. Denote by

$$Tf:=\sup_n |\sigma_n f| \qquad (f\in L^1)$$

the maximal function of the Fejér means of f. We prove that the operator T is of type (H, L^1).

THEOREM 5. $\|Tf\|_1\leqslant c\,\|f\|_H$ $(f\in H)$.

THEOREM 6. $|\{Tf>\lambda\}|\leqslant c\,\|f\|_1/\lambda$ *for all* $f\in L^1(I)$, $\lambda>\|f\|_1$, *and* $\|Tf\|_p\leqslant c_p\,\|f\|_p$ $(f\in L^p(I),\ 1<p\leqslant\infty)$. *That is, operator* T *is of weak type* $(1,1)$ *and of type* (p,p) *for each* $1<p\leqslant\infty$.

2. PROOFS

We need the following decomposition lemma of Calderon–Zygmund type.

LEMMA 4 ([Sch, CZ]). *Let* $f\in L^1(I)$, $\lambda>\|f\|_1$. *Then there exists a decomposition* $I=F\cup\bar{F}$ *such that*

$$F=\bigcup_{i=1}^{\infty} J_i$$

$$J_i\cap J_k=\varnothing\ (i\neq k),\qquad J_i=I_{k_i}(x^{(i)}),\ (k_i\in\mathbf{N},\ x^{(i)}\in I,\ i\in\mathbf{P}),$$

$$\|f_0\|_\infty<2\lambda,$$

$$\operatorname{supp} f_k\subseteq J_k,\qquad \int_{J_k} f_k=0,\qquad |J_k|^{-1}\int_{J_k}|f|\leqslant 4\lambda\qquad (k\in\mathbf{P}),$$

$$|F|\leqslant\|f\|_1/\lambda.$$

Proof of Lemma 3. Let $\mathbf{N}\ni n=\sum_{i=0}^{A} n_i 2^i$ $(n_A=1)$, $n^{(s)}:=n_A 2^A+\cdots+n_s 2^s$ $(s\leqslant A,\ s,\ A\in\mathbf{N})$. Set

$$K_{a,b}(z):=\sum_{k=a}^{b-1} D_k(z)\qquad a,\ b\in\mathbf{P},\ z\in I.$$

Suppose that $s>\tau$, $z\in I_\tau\backslash I_{\tau+1}$. By Lemma 2 we have

$$\begin{aligned}K_{n^{(s)},\,2^s}(z)&=\sum_{k=n^{(s)}}^{n^{(s)}+2^s-1} D_k(z)\\&=\sum_{k=n^{(s)}}^{n^{(s)}+2^s-1}\left(\sum_{j=0}^{\tau-1} k_j 2^j\right) v_k(z)+\sum_{k=n^{(s)}}^{n^{(s)}+2^s-1} k_\tau 2^\tau(-1)\,v_k(z)\\&=:\sum^{1}+\sum^{2}\end{aligned}$$

Since $2^s\mid n^{(s)}$, then $k=n^{(s)}+l$ $(0\leqslant l<2^s)$. We get $v_k(z)=v_{n^{(s)}+l}(z)=v_{n^{(s)}}(z)\,v_l(z)$. Thus $(\tau<s)$,

$$\sum^{1} = v_{n^{(s)}}(z) \sum_{k_{s-1}=0}^{1} \cdots \sum_{k_0=0}^{1} \left(\sum_{j=0}^{\tau-1} k_j 2^j \right) v_l(z)$$

$$= v_{n^{(s)}}(z) \sum_{k_{s-1}=0}^{1} \cdots \sum_{k_0=0}^{1} \left(\sum_{j=0}^{\tau-1} k_j 2^j \right) v_{k_0 2^0 + \cdots + k_{s-1} 2^{s-1}}(z)$$

$$= v_{n^{(s)}}(z) \sum_{k_{s-1}=0}^{1} \sum_{k_{s-2}=0}^{1} \cdots \sum_{k_{\tau+1}=0}^{1} \sum_{k_{\tau-1}=0}^{1} \cdots \sum_{k_0=0}^{1} \left(\sum_{j=0}^{\tau-1} k_j 2^j \right)$$

$$\times v_{k_0 2^0 + \cdots + k_{\tau-1} 2^{\tau-1} + k_{\tau+1} 2^{\tau+1} + \cdots + k_{s-1} 2^{s-1}}(z) \sum_{k_\tau=0}^{1} v_{k^\tau 2^\tau}(z)$$

$$= v_{n^{(s)}}(z)\, \phi(z) \sum_{k_\tau=0}^{1} v_{2^\tau}^{k_\tau}(z),$$

where $\phi(z)$ does not depend on k_τ. Since

$$v_{2^\tau}^{k_\tau}(z) = \varepsilon\left(k_\tau \left(\frac{z_\tau}{2} + \cdots + \frac{z_0}{2^{\tau+1}} \right) \right) = \varepsilon\left(\frac{k_\tau}{2} \right) = (-1)^{k_\tau},$$

we get $\sum^1 = 0$. Consequently,

$$|K_{n^{(s)},\, 2^s}(z)| = \left| \sum^{2} \right| = 2^{2^\tau} \left| \sum_{k_{s-1},\, \ldots,\, k_\tau} k_\tau \varepsilon \left(\sum_{j=\tau}^{s-1} k_j \left(\frac{z_j}{2} + \cdots + \frac{z_0}{2^{j+1}} \right) \right) \right|.$$

That is, $|K_{n^{(s)},\, 2^s}|$ does not depend on $n^{(s)}$. Thus,

$$\int_{I_\tau \setminus I_{\tau+1}} \sup_{\substack{2^{A+1} > N \geqslant 2^A \\ 2^s \mid N}} |K_{N,\, 2^s}|$$

$$= \frac{4^\tau}{2^s} \sum_{z_{\tau+1},\, \ldots,\, z_{s-1}} \left| \sum_{n_\tau,\, \ldots,\, n_{s-1}} n_\tau \varepsilon \left(\sum_{j=\tau}^{s-1} n_j \left(\frac{z_j}{2} + \cdots + \frac{z_0}{2^{j+1}} \right) \right) \right|$$

$$= \frac{4^\tau}{2^s} \sum_{z_{\tau+1},\, \ldots,\, z_{s-1}} \left| \sum_{n_{\tau+1},\, \ldots,\, n_{s-1}} \prod_{j=\tau+1}^{s-1} v_{2^j}^{n_j}(z) \right|$$

$$\leqslant \frac{4^\tau}{2^s} \left(\frac{2^s}{2^\tau} \right)^{1/2} \left\{ \sum_{z_{\tau+1},\, \ldots,\, z_{s-1}} \sum_{\substack{n_{\tau+1},\, \ldots,\, n_{s-1} \\ k_{\tau+1},\, \ldots,\, k_{s-1}}} \prod_{j=\tau+1}^{s-1} v_{2^j}^{n_j - k_j}(z) \right\}^{1/2} =: \sum^{3},$$

as we find with the help of the well-known Cauchy–Buniakovskii inequality:

$$v_{2^{s-1}}^{n_{s-1} - k_{s-1}}(z)$$

$$= \varepsilon\left((n_{s-1} - k_{s-1}) \frac{z_{s-1}}{2} \right) \varepsilon\left((n_{s-1} - k_{s-1}) \right) \left(\frac{z_{s-2}}{2^2} + \cdots + \frac{z_0}{2^s} \right) \Big).$$

Consequently,

$$\sum_{z_{s+1}=0}^{1} \sum_{\substack{n_{\tau+1}, \ldots, n_{s-1} \\ k_{\tau+1}, \ldots, k_{s-1}}} \prod_{j=\tau+1}^{s-1} v_{2^j}^{n_j - k_j}(z) = 2 \sum_{\substack{n_{\tau+1}, \ldots, n_{s-1} \\ k_{\tau+1}, \ldots, k_{s-1} \\ n_{s-1}=k_{s-1}}} \prod_{j=\tau+1}^{s-2} v_{2^j}^{n_j - k_j}(z).$$

Thus,

$$\sum^{3} = \frac{4^\tau}{2^s} \left(\frac{2^s}{2^\tau}\right)^{1/2} \left\{ 2 \sum_{z_{\tau+1}, \ldots, z_{s-2}} \sum_{\substack{n_{\tau+1}, \ldots, n_{s-1} \\ k_{\tau+1}, \ldots, k_{s-1} \\ n_{s-1}=k_{s-1}}} \prod_{j=\tau+1}^{s-2} v_{2^j}^{n_j - k_j}(z) \right\}^{1/2}$$

$$= \frac{4^\tau}{2^s} \left(\frac{2^s}{2^\tau}\right)^{1/2} \left\{ 2^2 \sum_{z_{\tau+1}, \ldots, z_{s-3}} \sum_{\substack{n_{\tau+1}, \ldots, n_{s-1} \\ k_{\tau+1}, \ldots, k_{s-1} \\ n_i = k_i (i = s-1, s-2)}} \prod_{j=\tau+1}^{s-3} v_{2^j}^{n_j - k_j}(z) \right\}^{1/2} = \cdots .$$

That is,

$$\sum^{3} \leqslant c \, \frac{4^\tau}{2^s} \left(\frac{2^s}{2^\tau}\right)^{1/2} (2^{s-\tau} 2^{s-\tau})^{1/2} \leqslant c (2^{s+\tau})^{1/2}.$$

That is, for $s > \tau$, we have

$$\int_{I_\tau \setminus I_{\tau+1}} \sup_{\substack{2^{A+1} > N \geqslant 2^A \\ 2^s \mid N}} |K_{N, 2^s}| \leqslant c \sqrt{2^{s+\tau}}. \tag{1}$$

By elementary calculation we have for $2^A \leqslant N < 2^{A+1}$ $(z \in I)$,

$$N K_N(z) = \sum_{s=0}^{A} N_s K_{N^{(s+1)}, 2^s}(z),$$

which gives the following inequality:

$$|N K_N(z)| \leqslant \sum_{s=0}^{A} |K_{N^{(s+1)}, 2^s}(z)| \qquad (2^A \leqslant N < 2^{A+1} z \in I).$$

Set $J^\tau := I_\tau \setminus I_{\tau+1}$.

$$\int_{J^\tau} \sup_{2^{A+1} > N \geqslant 2^A} |N K_N(z)| \, d\lambda(z)$$

$$\leqslant c \sum_{s=0}^{\tau} \int_{J^\tau} \sup_{2^{A+1} > N \geqslant 2^A} |K_{N^{(s+1)}, 2^s}| + \sum_{s=\tau+1}^{A} \int_{J^\tau} \sup_{2^{A+1} > N \geqslant 2^A} |K_{N^{(s+1)}, 2^s}|$$

$$=: \sum^{4} + \sum^{5}.$$

By Lemma 2 we have that for $z \in I_\tau \backslash I_{\tau+1}$, $D_n(z) \leqslant c 2^\tau$ for any $n \in \mathbf{N}$; thus $|K_{N^{(s+1)}, 2^s}(z)| \leqslant c 2^{s+\tau}$ for all N, $s \in \mathbf{N}$. This gives an upper bound for $\sum^4$:

$$\sum^4 \leqslant c \sum_{s=0}^{\tau} \frac{1}{2^{\tau+1}} 2^s 2^\tau \leqslant c 2^\tau.$$

The upper bound for $\sum^5$ is implied by (1) as follows:

$$\sum^5 \leqslant c \sum_{s=\tau+1}^{A} (2^{s+\tau})^{1/2}.$$

Consequently,

$$\int_{J^\tau} \sup_{2^{A+1} > N \geqslant 2^A} |K_N(z)| \, d\lambda(z) \leqslant c \frac{2^\tau}{2^A} + c \frac{(2^{A+\tau})^{1/2}}{2^A}.$$

This gives

$$\int_{J^\tau} \sup_{N \geqslant 2^A} |K_N(z)| \, d\lambda(z) \leqslant c \sum_{j=A}^{\infty} \left(\frac{2^\tau}{2^j} + c \frac{(2^{j+\tau})^{1/2}}{2^j} \right) \leqslant c 2^{(\tau - A)/2}.$$

Lemma 3 is proved. ∎

Proof of Theorem 6. It is known that $\|K_n\|_1 \leqslant c$ [S-W]. Consequently, operator T is of type (∞, ∞) (i.e., $\|Tf\|_\infty \leqslant c \|f\|_\infty$ for all $f \in L^\infty(I)$). We prove that T is of weak type (1, 1) (i.e., $|\{Tf > \lambda\}| \leqslant c \|f\|_1 / \lambda$ for all $f \in L^1(I)$, $\lambda > \|f\|_1$).

Let $\lambda > \|f\|_1$. Lemma 4 gives $\|\sigma_n f_0\|_\infty < c\lambda$,

$$\begin{aligned} &|\{x \in I : Tf(x) > 2c\lambda\}| \\ &\quad \leqslant |\{Tf_0 > c\lambda\}| + |F| + \left| \left\{ x \in I \backslash F : T\left(\sum_{j=1}^{\infty} f_j \right)(x) > c\lambda \right\} \right| \\ &\quad \leqslant c \|f\|_1/\lambda + \frac{1}{c\lambda} \int_{I \backslash F} \sum_{j=1}^{\infty} Tf_j \\ &\quad =: c \|f\|_1/\lambda + \frac{1}{c\lambda} \sum_{j=1}^{\infty} B^j. \end{aligned}$$

(Note that operator T is sublinear.)

$$I \backslash F \subset I \backslash J_j = \bigcup_{\tau=0}^{k_{j-1}} (I_\tau(x^{(j)}) \backslash I_{\tau+1}(x^{(j)})) =: \bigcup_{\tau=0}^{k_j - 1} J_j^\tau.$$

Denote by $\mathscr{A}_n$ the σ-algebra generated by the sets $I_n(x)$ $(x \in I)$ and by E_n the conditional expectation operator with respect to $\mathscr{A}_n$ $(n \in \mathbf{N})$. Since $\int_{I_{k_j}(x^{(j)})} f_j = 0$ and $\operatorname{supp} f_j \subseteq I_{k_j}(x^{(j)})$, then $E_{k_j} f_j = 0$ $(j \in \mathbf{P})$. That is, $n < 2^{k_j}$ implies $S_n f_j = S_n(E_{k_j} f_j) = 0$. Consequently, $Tf_j = \sup_{n \geqslant 2^{k_j}} |\sigma_n f_j|$. This gives

$$B^j \leqslant \sum_{\tau=0}^{k_j-1} \int_{J_j^\tau} \sup_{n \geqslant 2^{k_j}} \left| \int_{J_j} f_j K_n \right| =: \sum_{\tau=0}^{k_j-1} B_\tau^j,$$

$$B_\tau^j \leqslant \int_{J_j^\tau} \int_{J_j} |f_j(x)| \sup_{n \geqslant 2^{k_j}} |K_n(z-x)| \, d\lambda(z) \, d\lambda(x)$$

($-$ denotes the inverse of $+$). Lemma 3 gives $B_\tau^j \leqslant c f_{J_j} |f_j(x)| \, 2^{(\tau - k_j)/2} \, d\lambda(x)$, that is, $B^j \leqslant c \, \|f_j\|_1$ $(j \in \mathbf{P})$. This implies $|\{Tf > 2c\lambda\}| \leqslant c \, \|f\|_1/\lambda + (1/c\lambda) \sum_{j=1}^\infty \|f_j\|_1 \leqslant c \, \|f\|_1/\lambda$. Consequently, we proved that operator T is of weak type $(1, 1)$. Since T is of type (∞, ∞) and of weak type $(1, 1)$ then the interpolation theorem of Marczinkievicz [Sch] implies Theorem 6. ∎

Proof of Theorem 1. Since for a polynomial $P(x) = \sum_0^n c_k v_k(x)$ $(c_0, ..., c_n \in \mathbf{C}, n \in \mathbf{N}, x \in I)$ we have the relation $\sigma_n P(x) \to P(x)$ $(n \to \infty)$ for all $x \in I$ and since the set of polynomials is dense in the set of integrable functions in I, then by the usual density argument (see [Sim, Sch]) and by the weak $(1, 1)$ typeness of operator T follows the a.e. convergence $\sigma_n f \to f$ for all $f \in L^1(I)$. The proof of Theorem 1 is complete. ∎

Proof of Theorem 5. Let a be an atom ($a \neq 1$ can be supposed), $I_a := I_k(x)$, $\|a\|_\infty \leqslant 2^k$ for some $k \in \mathbf{N}$ and $x \in I$. Then $n < 2^k$ implies $S_n a = S_n E_k a = 0$. That is,

$$Ta = \sup_{n \geqslant 2^k} |\sigma_n a|.$$

Lemma 3 gives

$$\int_{I \setminus I_a} Ta = \sum_{j=0}^{k-1} \int_{I_j(x) \setminus I_{j+1}(x)} \sup_{n \geqslant 2^k} \left| \int_{I_k(x)} a(y) \, K_n(z-y) \, d\lambda(y) \right| d\lambda(z)$$

$$\leqslant \sum_{j=0}^{k-1} \int_{I_k(x)} |a(y)| \int_{I_j(x) \setminus I_{j+1}(x)} \sup_{n \geqslant 2^k} |K_n(z-y)| \, d\lambda(z) \, d\lambda(y)$$

$$\leqslant c \sum_{j=0}^{k-1} \int_{I_k(x)} |a(y)| \, (2^{j-k})^{1/2} \leqslant c \, \|a\|_1 \leqslant c.$$

Since Theorem 6 gives that operator T is of type (2, 2) (i.e., $\|Tf\|_2 \leqslant c\, \|f\|_2$ for all $f \in L^2(I)$), we have

$$\begin{aligned}\|Ta\|_1 &= \int_{I \setminus I_a} Ta + \int_{I_a} Ta \\ &\leqslant c + |I_a|^{1/2}\, \|Ta\|_2 \leqslant c + c2^{-k/2}\, \|a\|_2 \\ &\leqslant c + c2^{-k/2} 2^{k/2} \leqslant c.\end{aligned}$$

That is, $\|Ta\|_1 \leqslant c$ and consequently the sublinearity of T gives

$$\|Tf\|_1 \leqslant \sum_{i=0}^{\infty} |\lambda_i|\, \|Ta_i\|_1 \leqslant c \sum_{i=0} |\lambda_i| \leqslant c\, \|f\|_H$$

for all $f = \sum_{i=0}^{\infty} \lambda_i a_i \in H$. The proof of Theorem 5 is complete. ∎

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